Models of Set Theory I – Summer 2017

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Problem 17 [5 points] Let M be a countable and transitive model of ZFC, and let $\mathbb{P} = (P, \leq, 1_{\mathbb{P}}) \in M$ be a forcing notion. The *P-name space* of M is defined as follows. We say that $\dot{x}$ is a *P-name* if $\dot{x}$ is of the form $\dot{x} = \{(\dot{y}_i, p_i) \mid i \in I\}$ for some set I, such that each $\dot{y}_i$ is a P-name, and each p_i is an element of P. The P-name space of M is the collection of all P-names that are elements of M.

Verify the following properties of P-names.

- Whether or not some $\dot{x}$ is a P-name can be formulated as a first order property in the language of set theory, by a definite formula.
- If $\dot{x} \in M$, then there is a P-name $\dot{y} \in M$ such that $\dot{x}^G = \dot{y}^G$ whenever $G \subseteq P$ is a filter on $\mathbb{P}$. In fact, there is a map $F \colon M \to M$ which maps each $\dot{x} \in M$ to such an *equivalent* P-name $\dot{y}$, and the graph of F can be defined by a first order formula in the language of set theory within M.

Problem 18 [5 points] Let M be a countable and transitive model of ZFC.

1. Let $\mathbb{P}$ denote Cohen forcing. Show that $\mathbb{P} = (P, \leq, 1_{\mathbb{P}}) \in M$, and find a P-name $\dot{x} \in M$ such that for every $n \in \mathbb{N}$, there is a filter G on $\mathbb{P}$ such that $\dot{x}^G = n$, and such that $\dot{x}^G \in \mathbb{N}$ for every filter G on $\mathbb{P}$.
2. Let $\mathbb{P} = (P, \leq, 1_{\mathbb{P}}) \in M$ be an arbitrary forcing notion, and show that there cannot be a P-name $\dot{x} \in M$ such that for every $y \in M$ there is a filter G on $\mathbb{P}$ such that $\dot{x}^G = y$.

Problem 19 [4 points] Let M be a countable and transitive model of ZFC, and let $\mathbb{P} = (P, \leq, 1_{\mathbb{P}})$ denote Cohen forcing.

1. Show that whenever $x \subseteq \omega$, then x induces a filter
$$G_x = \{p \in P \mid p = x \restriction \mathrm{dom}\ p\}.$$
2. Show that there is a filter G on $\mathbb{P}$ such that $M[G]$ does not satisfy all axioms of ZFC.

Hint: Since the *ordinal height* α of M (that is $\alpha = M \cap \mathrm{Ord}$) is countable, we find a wellordering $\prec$ of ω of order-type α. Using a bijection between $\omega \times \omega$ and ω, we may *code* $\prec$ by some $x \subseteq \omega$. Make these remarks precise, and then show that $M[G_x]$ cannot satisfy all axioms of ZFC.

Problem 20 [6 points] A *Boolean Algebra* is a set B with two binary operations $\wedge$ and $\vee$, a unary operation $\neg$ and two elements 0 and 1, satisfying the following axioms, for all $a, b, c \in B$.

$a \vee (b \vee c) = (a \vee b) \vee c$	$a \wedge (b \wedge c) = (a \wedge b) \wedge c$	associativity
$a \vee b = b \vee a$	$a \wedge b = b \wedge a$	commutativity
$a \vee (a \wedge b) = a$	$a \wedge (a \vee b) = a$	absorption
$a \vee 0 = a$	$a \wedge 1 = a$	identity
$a \wedge 0 = 0$	$a \vee 1 = 1$	extremality
$a \vee a = a$	$a \wedge a = a$	idempotence
$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$	$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$	distributivity
$a \vee \neg a = 1$	$a \wedge \neg a = 0$	complements

One can define a natural ordering on a Boolean algebra B by setting, for $a, b \in B$,

$$a \leq b \iff a \wedge b = a.$$

Let B be the domain of a Boolean algebra with the above operations and ordering, let $B^* = B \setminus \{0\}$, and let $\mathbb{B} = (B^*, \leq, 1)$. Verify the following.

- For all $a, b \in B$, $a \leq b \iff a \vee b = b$.
- $\mathbb{B}$ is a forcing notion.
- B^* is *separative*, that is for $p, q \in B^*$, if $\neg(p \leq q)$, then there is $r \leq p$ in B^* such that $r \wedge q = 0$.
- If $\mathbb{P} = (P, \leq_{\mathbb{P}}, 1_{\mathbb{P}})$ is a partial order, consider the following equivalence relation $\sim$ on P. We say that for $p, q \in P$, $p \sim q$ if and only if

  $$\forall r \; [r \text{ is compatible with } p \iff r \text{ is compatible with } q].$$

  We define the *separative quotient* of $\mathbb{P}$ to be the following partial order $\mathbb{Q} = (Q, \leq_{\mathbb{Q}}, 1_{\mathbb{Q}})$. $Q = P/\sim\; = \{[p]_\sim \mid p \in P\}$. For $[p]_\sim, [q]_\sim \in Q$, we let $[p]_\sim \leq [q]_\sim$ if and only if

  $$\forall r \leq p \; [r \text{ and } q \text{ are compatible}].$$

  Show that Q is a well-defined separative partial order with the following properties

  - $p \leq q$ implies $[p]_\sim \leq [q]_\sim$, and
  - p and q are compatible in $\mathbb{P}$ if and only if $[p]_\sim$ and $[q]_\sim$ are compatible in $\mathbb{Q}$.